

LOW COHOMOGENEITY AND POLAR ACTIONS ON EXCEPTIONAL COMPACT LIE GROUPS

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ABSTRACT. We study isometric Lie group actions on the compact exceptional groups E_6 , E_7 , E_8 , F_4 and G_2 endowed with a biinvariant metric. We classify polar actions on these groups, in particular, we show that all polar actions are hyperpolar. We determine all isometric actions of cohomogeneity less than three on E_6 , E_7 , F_4 and all isometric actions of cohomogeneity less than 20 on E_8 . Moreover we determine the principal isotropy algebras for all isometric actions on G_2 .

INTRODUCTION

We study isometric Lie group actions on the compact exceptional simple Lie groups E_6 , E_7 , E_8 , F_4 and G_2 endowed with a biinvariant Riemannian metric; we classify actions with low cohomogeneity and polar actions on these spaces. An isometric action of a compact Lie group on a Riemannian manifold is called *polar* if there exists an immersed connected submanifold Σ which intersects the orbits orthogonally and meets every orbit. Such a submanifold Σ is called a *section* of the group action. If the section is flat in the induced metric, the action is called *hyperpolar*. Our main result is the following.

Theorem 1. *Let L be a connected simple compact Lie group of type E_6 , E_7 , E_8 , F_4 or G_2 endowed with a biinvariant Riemannian metric. Let $H \subseteq L \times L$ be a closed subgroup such that the action of H on L defined as in (1) is polar. Then the H -action on L is hyperpolar or the H -orbits are finite.*

Moreover, if the cohomogeneity of the polar H -action on L is greater than two, then H is a symmetric subgroup of $L \times L$.

In the course of proving Theorem 1, we obtain an explicit description of all polar actions of connected groups on the exceptional compact Lie groups. As a further result, we classify actions of certain low cohomogeneities on the exceptional groups, cf. Theorems 18, 19, 17, 20.

It should be noted that the classification problem for polar actions in the special case that the section is flat, i.e. for hyperpolar actions, had been solved before. In fact, the author has classified hyperpolar actions on all irreducible compact symmetric spaces in [18].

If the additional assumption that the section is flat is dropped, i.e. if one considers actions on irreducible compact symmetric spaces which are polar, but not necessarily hyperpolar, then there is a sharp contrast between the case of rank-one symmetric spaces and the higher rank case; while there are many examples of polar

actions with non-flat sections on rank-one symmetric spaces, see [25] for a classification, there are as yet no examples known on the spaces of higher rank. In fact, there is the following conjecture.

Conjecture 2 (Biliotti [2]). *Any polar action with orbits of positive dimension on an irreducible compact Riemannian symmetric space of higher rank is hyperpolar.*

This conjecture was shown to be true for all symmetric spaces of type I, i.e. symmetric spaces G/K where G is a simple compact Lie group and K is a symmetric subgroup, by the author in [19]. Earlier the conjecture had been proved to hold for actions with a fixed point by Brück [4], for actions on the complex quadrics by Podestà and Thorbergsson [24], on complex Grassmannians by Biliotti and Gori [3], and by Biliotti [2] for compact irreducible Hermitian symmetric spaces. It remained open for the case of symmetric spaces of type II, i.e. the simple compact connected Lie groups equipped with a biinvariant metric.

Our Theorem 1 now confirms Biliotti's conjecture in the special case of exceptional compact Lie groups. However, the conjecture still remains open for polar actions on the classical Lie groups $SO(n)$, $SU(n)$ and $Sp(n)$.

A prominent special case of (hyper)polar actions of independent interest, which has been studied by many authors, is the case of *cohomogeneity one actions*, i.e. such actions where the principal orbits are hypersurfaces. Hsiang and Lawson [16], Takagi [23], D'Atri [6], and Iwata [17] have determined all cohomogeneity one actions on S^n , \mathbb{CP}^n , \mathbb{HP}^n and \mathbb{OP}^2 , respectively.

In [18] the author has classified cohomogeneity one actions on all irreducible compact symmetric spaces, in particular, on simple compact Lie groups. However, the classification there is only up to orbit equivalence. Theorem 18 is therefore a refinement of this classification in that all connected closed subgroup of the isometry group are given which act with cohomogeneity one.

Motivated by the interest in cohomogeneity one actions, we carry on the study of actions whose principal orbits have low codimension in this article and classify actions of cohomogeneity two on the exceptional compact Lie groups, cf. Theorem 19.

For the groups G_2 and E_8 , we can further improve these results. It turns out that with few exceptions, given by Theorem 17, all isometric actions on G_2 have finite principal isotropy groups and hence we have, in particular, determined the (co)dimensions of the principal orbits of all isometric actions on G_2 . Finally, we classify all isometric actions on E_8 of cohomogeneity less than 20, see Theorem 20.

This article is organized as follows. Theorem 1 is proved in Sections 2–10. The rest of the article is concerned with actions of low cohomogeneity. In Section 11 we determine the Lie algebra type of the principal isotropy subgroups for every isometric action of a compact Lie group on G_2 . In Section 12, we determine all isometric actions of compact Lie groups on the exceptional groups where the cohomogeneity is less than three. Since these actions occur as candidates for polar actions, we can use the proof of Theorem 1 to a large extent. In Section 13 we classify low cohomogeneity actions on E_8 .

1. PRELIMINARIES

In this article, our objects of study are simple compact connected Lie groups L , endowed with a biinvariant Riemannian metric. Such a metric is unique up to a constant scaling factor, whose choice is of course irrelevant here; we may for instance

assume that L is equipped with the homogeneous metric induced by the negative of the Killing form.

Now let H be a compact Lie group acting isometrically on L . The action is polar if and only if the action restricted to the connected component of H is polar and furthermore the cohomogeneity of the H -action remains the same if the action is restricted to the connected component of H . Therefore we will assume that H is a closed connected subgroup of $L \times L$ (which covers the connected component of the isometry group of L) and that the action of H on L is given by

$$(1) \quad (h_1, h_2) \cdot \ell = h_1 \ell h_2^{-1} \text{ for } (h_1, h_2) \in H, \ell \in L.$$

Assume the groups H_1 and H_2 act isometrically on the Riemannian manifolds M_1 and M_2 , respectively. The H_1 -action on M_1 and the H_2 -action on M_2 are called *conjugate* if there exists an isometry $F: M_1 \rightarrow M_2$ and an isomorphism $\phi: H_1 \rightarrow H_2$ such that $F(g \cdot p) = \phi(g) \cdot F(p)$ for all $g \in H$, $p \in M_1$. For the purposes of this article, it obviously suffices to consider actions up to conjugacy.

Let L be a semisimple compact Lie group equipped with the biinvariant metric induced by the negative of the Killing form and let H be a closed subgroup of $L \times L$. Then any automorphism $\sigma: L \rightarrow L$ is an isometry and the H -action on L is conjugate to the action of $\phi(H)$ on L where $F = \sigma$ and $\phi(h_1, h_2) = (\sigma(h_1), \sigma(h_2))$. Let $\ell, r \in L$, then the map $L \rightarrow L$, $F: g \mapsto \ell g r^{-1}$ is an isometry of L and the H -action is conjugate to the action of $\{(\ell h_1 \ell^{-1}, r h_2 r^{-1}) \mid (h_1, h_2) \in H\}$. Furthermore, the map $F: L \rightarrow L$, $g \mapsto g^{-1}$ is an isometry and the H -action on L is conjugate to the action of $\{(h_2, h_1) \in L \times L \mid (h_1, h_2) \in H\}$. However, notice that if α is an outer automorphism of L , then the action of $\{(h_1, \alpha(h_2)) \in L \times L \mid (h_1, h_2) \in H\}$ is in general not conjugate to the H -action on L , see [18], Theorem 3.2 and the preceding remarks.

Notation. Since in this article we frequently have to deal with certain subgroups of $L \times L$, where L is a simple compact Lie group, it is convenient to adopt the following notational convention: If H_1 and H_2 are subgroups of L , then $H_1 \times H_2$ *always* denotes the subgroup $\{(h_1, h_2) \mid h_1 \in H_1, h_2 \in H_2\}$ of $L \times L$ and we avoid the use the symbol “ \times ” altogether whenever we consider direct products of groups otherwise. If K is a subgroup of L , we denote by ΔK the diagonally embedded subgroup $\{(g, g) \mid g \in K\}$ of $L \times L$. More generally, if K is a subgroup of L and $\sigma: L \rightarrow L$ is an automorphism of L , then $\Delta^\sigma K$ stands for the subgroup $\{(g, \sigma(g)) \mid g \in K\}$ of $L \times L$. Whenever we consider a closed subgroup H of $L \times L$ given by one of the three notations just described above, it is henceforth always understood that the action of H on L is given by (1). In some cases we use the notation of [7] to uniquely describe the conjugacy class of a subgroup in an exceptional compact Lie group, e.g. in Table 7, we denote by G_2^1 and G_2^3 two non-conjugate subgroups of E_6 which have Dynkin index 1 and 3, respectively.

If $H_1 \subset L$ is a closed subgroup such that its Lie algebra is the fixed point set of an involutive automorphism of the Lie algebra of L , then we call H_1 a *symmetric subgroup* of L . If $H = H_1 \times H_2$ where $H_1, H_2 \subset L$ are symmetric subgroups, then the H -action on L is called a *Hermann action* [14]. The action of $\Delta^\sigma L$ on L is called the σ -action on L . Hermann actions and σ -actions are well known to be hyperpolar [12].

If H is a closed connected subgroup of $L \times L$ and H' is a closed subgroup of H then we will refer to the action of H' on L as a *subaction* of the H -action. If

in addition the H' -action and the H -action on L are orbit equivalent, we say the H' -action is an *orbit equivalent subaction* of the H -action.

Let L be a simple compact connected Lie group and let \mathfrak{l} be its Lie algebra. Let H be a closed connected subgroup of $L \times L$. Let $\pi_1, \pi_2: \mathfrak{l} \times \mathfrak{l} \rightarrow \mathfrak{l}$ be the canonical projections such that $\pi_1(X, Y) = X$ and $\pi_2(X, Y) = Y$. Let $\mathfrak{h}_1 = \ker \pi_2|_{\mathfrak{h}}$, $\mathfrak{h}_2 = \ker \pi_1|_{\mathfrak{h}}$. Since \mathfrak{h} is reductive, there exists a complementary ideal \mathfrak{h}_3 of $\mathfrak{h}_1 + \mathfrak{h}_2$ in \mathfrak{h} . Since $\mathfrak{h}_1 \cap \mathfrak{h}_2 = 0$, the Lie algebra \mathfrak{h} of H is isomorphic to the direct sum $\mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \mathfrak{h}_3$ of Lie algebras.

Lemma 3. *Let L be a simple compact connected Lie group and let H be a closed connected subgroup of $L \times L$ such that the H -action on L is not transitive. Then H is contained in at least one of the following subgroups of $L \times L$.*

- (i) $\Delta^\sigma L$, where $\sigma: L \rightarrow L$ is an automorphism of L .
- (ii) $H' \times H''$, where $H', H'' \subset L$ are maximal connected subgroups of L .

Proof. With the notation as above, consider $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \mathfrak{h}_3$. Assume that $\mathfrak{h}_1 \cong \mathfrak{l}$. Then it follows that H contains the subgroup $L \times \{1\}$, which acts transitively on L , a contradiction. The same argument shows that $\mathfrak{h}_2 \not\cong \mathfrak{l}$. Assume $\mathfrak{h}_3 \cong \mathfrak{l}$. Since $\pi_1|_{\mathfrak{h}_1 \oplus \mathfrak{h}_3}$ and $\pi_2|_{\mathfrak{h}_2 \oplus \mathfrak{h}_3}$ are injective, it follows that $\mathfrak{h}_1 = \mathfrak{h}_2 = \{0\}$ and that $\pi_1|_{\mathfrak{h}_3}$ and $\pi_2|_{\mathfrak{h}_3}$ are Lie algebra isomorphisms $\mathfrak{h}_3 \rightarrow \mathfrak{l}$. It follows that $\mathfrak{h} = \{(\pi_1(X), \pi_2(X)) \mid X \in \mathfrak{h}_3\}$ and thus $H = \Delta^\sigma L$, where $(\sigma_*)_{\mathfrak{e}} = \pi_2|_{\mathfrak{h}_3} \circ \pi_1|_{\mathfrak{h}_3}^{-1}$. Now assume $\mathfrak{h}_3 \not\cong \mathfrak{l}$. It follows that $\mathfrak{h} \subseteq \pi_1(\mathfrak{h}) \times \pi_2(\mathfrak{h})$, where $\pi_i(\mathfrak{h})$ are proper subalgebras of \mathfrak{l} . \square \square

The maximal connected subgroups of compact Lie groups are classified in [7] and [8], cf. also [18], Theorems 2.1 and 2.2. There is the following criterion for polarity of an isometric action on a symmetric space. For a proof see [10] or [18]. Note that sections of polar actions on Riemannian manifolds are always totally geodesic submanifolds.

Proposition 4. *Let G be a connected compact Lie group, let $K \subset G$ be a symmetric subgroup and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition. Let $H \subseteq G$ be a closed subgroup. Let k be the cohomogeneity of the H -action on G . Then the following are equivalent.*

- (i) *The H -action on G/K is polar w.r.t some Riemannian metric induced by an $\text{Ad}(G)$ -invariant scalar product on \mathfrak{g} .*
- (ii) *For any $g \in G$ such that gK lies in a principal orbit of the H -action on G/K the subspace $\nu = g^{-1}N_{gK}(H \cdot gK) \subseteq \mathfrak{p}$ is a k -dimensional Lie triple system such that the Lie algebra $\mathfrak{s} = \nu \oplus [\nu, \nu]$ generated by ν is orthogonal to $\text{Ad}(g^{-1})\mathfrak{h}$.*
- (iii) *The normal space $N_{\mathfrak{e}K}(H \cdot \mathfrak{e}K) \subseteq \mathfrak{p}$ contains a k -dimensional Lie triple system ν such that the Lie algebra $\mathfrak{s} = \nu \oplus [\nu, \nu]$ generated by ν is orthogonal to \mathfrak{h} .*

Remark. Hyperpolar actions are characterized by the additional property that the Lie triple system ν in Proposition 4 is abelian (in which case the Lie algebra \mathfrak{s} is equal to ν). In case the H -action on G/K is polar, the Lie triple system ν corresponds to the tangent space of a section containing $\mathfrak{e}K$. Note that cohomogeneity one actions are hyperpolar.

Let us now apply the criterion from Proposition 4 to the case of a compact Lie group L equipped with a biinvariant metric. To this end, we present the symmetric

space L homogeneously as $G/K = (L \times L)/\Delta L$. The Lie algebra of $K = \Delta L$ is

$$(2) \quad \mathfrak{k} = \{(X, X) \mid X \in \mathfrak{l}\},$$

where \mathfrak{l} denotes the Lie algebra of L . The Cartan complement of \mathfrak{k} in $\mathfrak{g} \cong \mathfrak{l} \oplus \mathfrak{l}$ is

$$(3) \quad \mathfrak{p} = \{(X, -X) \mid X \in \mathfrak{l}\}.$$

Assume the subgroup $H \subset G$ is of the form $H = H_1 \times H_2$, where H_1 and H_2 are closed subgroups of L , i.e.

$$(4) \quad H = \{(h_1, h_2) \mid h_1 \in H_1, h_2 \in H_2\}.$$

Let \mathfrak{m}_1 and \mathfrak{m}_2 be the orthogonal complements of \mathfrak{h}_1 and \mathfrak{h}_2 , respectively in the Lie algebra \mathfrak{l} of L . By conjugation of H we may assume without loss of generality that the identity element e of G lies in a principal orbit of the H -action on G/K . Then the subspace $\nu \subset \mathfrak{g}$ in Proposition 4 (ii) is given by

$$(5) \quad \nu = \{(X, -X) \mid X \in \mathfrak{m}_1 \cap \mathfrak{m}_2\}$$

and $[\nu, \nu]$ is spanned by the elements

$$([X, Y], [X, Y]), \quad X, Y \in \mathfrak{m}_1 \cap \mathfrak{m}_2.$$

If now, say, H_2 is a symmetric subgroup of L , then we have from the Cartan relations $[\mathfrak{m}_2, \mathfrak{m}_2] \subseteq \mathfrak{h}_2$. Thus if $[\nu, \nu] \neq 0$ then it follows that $[\nu, \nu]$ is not orthogonal to \mathfrak{h} . We have shown:

Lemma 5. *Let L be a compact Lie group equipped with a biinvariant metric. Let $H = H_1 \times H_2 \subset L \times L$ be closed subgroup as in (4) and such that $H_2 \subset L$ is a symmetric subgroup. Then the action of H on L is polar if and only if it is hyperpolar.*

The following Theorem was shown in [19], Theorem 5.4

Theorem 6. *If a compact connected Lie group acts polarly and non-trivially on an irreducible compact symmetric space then every section is covered by a Riemannian product of spaces which have constant curvature.*

Lemma 7. *Let L be a simple compact Lie group and let $H \subset L \times L$ be closed subgroup action polarly on L . Then $\dim H \geq \dim L - 3 \cdot \text{rk} L$. In particular, a compact connected nontrivial Lie group acting polarly on one of the exceptional groups G_2, F_4, E_6, E_7, E_8 is of dimension greater or equal 8, 40, 60, 112, 224, respectively.*

Proof. This is an immediate consequence of [19], Lemma 3.3. \square \square

Lemma 8. *Let L be an exceptional simple compact Lie group F_4, E_6, E_7 , or E_8 and let*

$$K = \{(k_1, k_2) \mid k_1 \in K_1, k_2 \in K_2\} \subseteq L \times L,$$

where K_1 and K_2 are closed proper subgroups of L . If a closed subgroup of K acts polarly on L , then $\dim K_i \geq 12, 15, 34, 90$, respectively.

Proof. Let $H \subseteq K$ be a closed connected subgroup acting polarly on L . Let $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \mathfrak{h}_3$ be the Lie algebra of H with the notation as in Lemma 3. Then $\pi_2(\mathfrak{h}) \cong \mathfrak{h}_2 \oplus \mathfrak{h}_3$ is a subalgebra in the Lie algebra of K_2 . Assume \mathfrak{h}_2 corresponds to a symmetric subgroup of L . Then $\pi_2(\mathfrak{h}_2)$ a maximal subalgebra of \mathfrak{l} and it follows that $\mathfrak{h}_3 = 0$. Thus the H -action is hyperpolar by Lemma 5 in this case and it follows that $\dim K_1 \geq 12, 20, 47, 104$, respectively, see [18], Section 2.4.4.

Now assume \mathfrak{h}_2 does not correspond to a symmetric subgroup of L . We determine non-symmetric connected subgroups of maximal dimension in the groups F_4 , E_6 , E_7 and E_8 . Such groups are $\text{Spin}(8) \subset F_4$, $\text{Spin}(10) \subset E_6$, $E_6 \subset E_7$ and $E_7 \cdot U(1) \subset E_8$, see [7], cf. also Tables 6, 7 and 8. Thus the maximal dimension of a proper closed non-symmetric subgroup in F_4 , E_6 , E_7 , or E_8 is 28, 45, 78 and 134, respectively and the assertion of the Lemma now follows directly from Lemma 7, since $\pi_1(\mathfrak{h}) \cong \mathfrak{h}_1 \oplus \mathfrak{h}_3$ is a subalgebra in the Lie algebra of K_1 . \square \square

Assume a Lie group G acts isometrically on a Riemannian manifold M and let $p \in M$. Then the *isotropy subgroup* $G_p = \{g \in G \mid g \cdot p = p\}$ acts on $T_p M$ such that the tangent space $T_p(G \cdot p)$ and the normal space $N_p(G \cdot p)$ to the G -orbit through p are invariant subspaces. The action of G_p on $N_p(G \cdot p)$ is called the *slice representation* of the G -action at p . The slice representation is trivial if and only if p lies in a principal orbit. Slice representations are an import tool for analyzing Lie group actions on manifolds since they provide a local linearization of the group action. In fact, they will be our main tool in this article. A slice representation has the same cohomogeneity as the action of G on M ; if the action of G on M is polar, then all slice representations are polar, too; however, the converse is not true, see Section 12.

2. POLARITY OF SUBACTIONS

For hyperpolar actions on symmetric spaces of the compact type one has the maximality property that the action of a closed subgroup H of the isometry group can only have hyperpolar subactions if the H -action is itself hyperpolar (maybe transitive); this follows immediately from Proposition 4. In fact, in case the symmetric space is irreducible, the following stronger statement holds: If there is an inclusion relation between two closed subgroups of the isometry group which both act hyperpolarly then the actions are orbit equivalent (see below) or one action is transitive, see Corollary D of [11]. Therefore, it is sufficient to consider only maximal nontransitive subgroups of the isometry group in order to find all groups acting hyperpolarly on a given irreducible compact symmetric space, cf. the classification in [18].

However, for polar actions such a maximality property does not hold in general; for example, let G_1, G_2 be compact Lie groups acting orthogonally on \mathbb{R}^{n_1} and \mathbb{R}^{n_2} , respectively, such that the G_1 -action is polar, but the G_2 -action is not. Then the action of the direct product of G_1 and G_2 on $S^{n_1+n_2-1} \subset \mathbb{R}^{n_1} \oplus \mathbb{R}^{n_2}$ is not polar, even though the action restricted to the subgroup G_1 is. To overcome this difficulty, we introduce the notion of polarity minimality, which means that an action does not have any polar subactions in a nontrivial way.

We say that an action of a group G on a Riemannian manifold M is *orbit equivalent* to the action of a group G' on a Riemannian manifold M' if there is an isometry $F: M \rightarrow M'$ such that F maps any connected component of a G -orbit in M onto a connected component of a G' -orbit in M' .

Definition 1. Let G be a compact Lie group acting isometrically on a Riemannian manifold. We say the action of G on M is *polarity minimal* if any closed connected subgroup of G whose action on M is nontrivial and not orbit equivalent to the G -action is non-polar.

Note that a polarity minimal action can be polar or non-polar. We cite the following proposition from [19], which gives some sufficient conditions for an orthogonal representation to be polarity minimal.

Proposition 9. *Let $\rho: G \rightarrow \mathrm{O}(V)$ be a representation of the compact connected Lie group G . Then ρ is polarity minimal if one of the following holds.*

- (i) *The representation ρ is irreducible of cohomogeneity ≥ 2 .*
- (ii) *The representation space V is the direct sum of two equivalent G -modules.*
- (iii) *The representation space V contains a G -invariant submodule W such that the G -representation on W is almost effective, non-polar, and polarity minimal.*

We will use Proposition 9 to show that in many cases an action on an exceptional group has a polarity minimal slice representation. Under various conditions, some of which are collected in the following proposition, this is sufficient to show that the action under consideration is polarity minimal itself. This will be the main tool of our classification.

Lemma 10. *Let G be compact Lie group and $K \subset G$ be symmetric subgroup such that $M = G/K$ is an irreducible symmetric space and let $H \subset G$ be a closed subgroup. The action of H on M is non-polar and polarity minimal if there is a non-polar polarity minimal submodule $V \subseteq \mathrm{N}_p(H \cdot p)$ of the slice representation at p such that one of the following holds.*

- (i) *M is Hermitian symmetric and $\dim(V) > \mathrm{rk}(H)$.*
- (ii) *$\dim(V) > s(M)$, where $s(M)$ is the maximal dimension of a totally geodesic submanifold of M locally isometric to a product of spaces with constant curvature, cf. [19], Lemma 3.3.*
- (iii) *$V \subseteq \mathfrak{p} = \mathrm{T}_p M$ (where $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ as usual such that \mathfrak{k} is the Lie algebra of $K = G_p$) contains a Lie triple system corresponding to an irreducible symmetric space of nonconstant curvature, e.g. an irreducible symmetric space of higher rank.*
- (iv) *The isotropy group $H \cap K$ acts almost effectively on V and $\mathrm{rk}(H \cap K) = \mathrm{rk}(H)$.*

Proposition 11. *Let M be a simple compact Lie group and let H be a closed connected subgroup of $M \times M$ which acts hyperpolarly and with cohomogeneity $k \geq 2$ on M . Let $H' \subset H$ be a closed subgroup acting on M with cohomogeneity $k + 1$. Then the H' -action on M is not polar.*

Proof. Assume the H' -action on M is polar. By the results of [18], we may assume that the H -action is a σ -action or a Hermann action. By Section 3.2 of [18] and Table 5 of [19] we know that there is a point $p \in M$ such that the slice representation of H_p on $\mathrm{N}_p(H \cdot p)$ is irreducible. There are two alternatives for the slice representation of H'_p on $\mathrm{N}_p(H' \cdot p)$: Either $\mathrm{N}_p(H \cdot p) = \mathrm{N}_p(H' \cdot p)$ and H'_p acts with cohomogeneity $k + 1$ on $\mathrm{N}_p(H \cdot p)$ or $\mathrm{N}_p(H \cdot p)$ is a proper submodule of $\mathrm{N}_p(H' \cdot p)$ on which H'_p acts with cohomogeneity k , hence irreducibly. In the first case a contradiction arises with Proposition 9 (i). Now consider the second case. Let Σ' be a section of the H' -action on M such that $p \in \Sigma'$. It follows from Corollary D of [11] that Σ' is not flat. Since H'_p acts irreducibly on $\mathrm{N}_p(H \cdot p)$, the Weyl group of the slice representation of H'_p acts irreducibly on the hyperplane

$T_p \Sigma' \cap N_p(H \cdot p)$ in $T_p \Sigma'$. Thus Σ' is an irreducible symmetric space of rank $k \geq 2$ and dimension $k + 1$ and we have reached a contradiction. \square \square

In [9] a list of orbit equivalent subactions of irreducible polar representations of cohomogeneity ≥ 2 is given, we reproduce this list in Table 1 for the convenience of the reader. Table 1 is to be interpreted as follows: If a connected compact Lie

G	K	K'	Condition
SO(9)	SO(7) · SO(2)	$G_2 \cdot \text{SO}(2)$	
SO(10)	SO(8) · SO(2)	Spin(7) · SO(2)	
SO(11)	SO(8) · SO(3)	Spin(7) · SO(3)	
SU($p + q$)	S(U(p) · U(q))	SU(p) · SU(q)	$p \neq q$
SO($2n$)	U(n)	SU(n)	n odd
E_6	Spin(10) · U(1)	Spin(10)	

TABLE 1. Orbit equivalent subactions of polar representations.

group K' acts on some finite dimensional Euclidean vector space by an irreducible polar representation such that the action is non-transitive on the unit sphere, then either the K' -representation is equivalent to an isotropy representation of a Riemannian symmetric space, or there is a Riemannian symmetric space G/K and the K' -representation is equivalent to the isotropy representation of G/K restricted to the subgroup $K' \subset K$ where the triple (G, K, K') is as in Table 1. In the latter case, the K -action and the K' -action are orbit equivalent.

3. SUBACTIONS OF σ -ACTIONS

In this section, we do not restrict ourselves to the case of exceptional groups; we will show for any connected simple compact Lie group of rank greater than one that for any closed subgroup of $\Delta^\sigma L$ acting polarly on L the sections Σ are either flat or $\Sigma = L$.

Let L be a simply connected simple compact Lie group with $\text{rk} L \geq 2$, equipped with a biinvariant Riemannian metric. Let σ be an automorphism of L and let $\Delta^\sigma L \subset L \times L$ be a subgroup of the form

$$\Delta^\sigma L = \{(g, \sigma(g)) \mid g \in L\}.$$

In this section, we will consider the action of $\Delta^\sigma L$ and of closed connected subgroups $H \subset \Delta^\sigma L$ on L . We may assume that σ is either the identity or is induced by a nontrivial automorphism of the Dynkin diagram of L .

In the first case, the action of $\Delta^{\text{id}_L} L = \Delta L$ is simply the adjoint action of L . In particular, the identity element of L is a fixed point and it follows by Corollary 6.2 of [18], cf. also [4], Theorem 2.2, that the action of any closed connected subgroup $H \subset \Delta L$ on L is hyperpolar and in fact orbit equivalent to the ΔL -action since $\text{rk} L \geq 2$. This implies $H = \Delta L$ by Table 1.

Now let σ be an outer automorphism of L induced by an automorphism of the Dynkin diagram of L . Then L , the order of σ , the connected component of the fixed point group L^σ and the cohomogeneity of the $\Delta^\sigma L$ -action on L are as given by Table 2, cf. [18], Theorem 3.4 and [13], Ch. X, §5.

L	$SU(2n+1)$	$SU(2n)$	$SO(2n)$	E_6	$Spin(8)$
$\text{ord}(\sigma)$	2	2	2	2	3
$(L^\sigma)_0$	$SO(2n+1)$	$Sp(n)$	$SO(2n-1)$	F_4	G_2
Cohomogeneity	n	n	$n-1$	4	2

TABLE 2. σ -Actions where σ is an outer automorphism.

As shown in Section 3.2 of [18], the normal space to the Δ^σ -orbit at the identity element of L is

$$(6) \quad \{(X, -X) \mid X \in \mathfrak{l}, (\sigma_*)_e(X) = X\} \subset \mathfrak{p},$$

where $(\sigma_*)_e : \mathfrak{l} \rightarrow \mathfrak{l}$ denotes the differential of σ at the identity element of L ; furthermore, the slice representation at $e \in L$ is equivalent to the adjoint representation of the fixed point group $L^\sigma = \{g \in L \mid \sigma(g) = g\}$. Assume $H \subset \Delta^\sigma L$ is a closed connected subgroup acting polarly on L .

First assume $\text{rk}(L^\sigma) \geq 2$. Consider the isotropy subgroup H_e of the H -action at e . Its slice representation contains (6) as a submodule. Since $\text{rk}(L^\sigma) > 1$, it follows from Proposition 9 (i) that either the action of H_e on (6) is orbit equivalent to the action of $(\Delta^\sigma L)_e$ or the subspace (6) of $\mathfrak{p} = T_e L$ is tangent to a section through e , contradicting Theorem 6, since (6) is an irreducible Lie triple system of higher rank. Thus it follows that the actions of H_e and $(\Delta^\sigma L)_e$ on (6) are orbit equivalent. Since the slice representation of $(\Delta^\sigma L)_e = \Delta L^\sigma \cong L^\sigma$ is equivalent to the adjoint representation of L^σ , one can see from Table 1 that $(H_e)_0 = ((\Delta^\sigma L)_e)_0$. Since this argument can be applied to all conjugate subgroups $gHg^{-1} \subset \Delta^\sigma L$, $g \in \Delta^\sigma L$, it follows that H contains all conjugates of $((\Delta^\sigma L)_e)_0$. Since $\Delta^\sigma L$ is a simple connected Lie group, this shows that $H = \Delta^\sigma L$.

If $\text{rk}(L^\sigma) = 1$ then $L \cong SU(3)$ and $\sigma : L \rightarrow L$ is given by complex conjugation. Consider the action of $\Delta^\sigma S(U(2) \cdot U(1))$ on $SU(3)$; it has a slice representation where a one-dimensional isotropy group acts nontrivially on two two-dimensional submodules; this representation is non-polar. Any other subgroups of $SU(3)$ are of dimension ≤ 3 ; however it is easy to see that $SU(3)$ does not contain any totally geodesic subspaces of dimension ≥ 5 locally isometric to a product of spaces of constant curvature. We have shown the following.

Proposition 12. *Let L be simple compact connected Lie group of rank greater than one and let σ be an automorphism of L . Assume the action of a closed connected non-trivial subgroup H of $\Delta^\sigma L$ on L is polar. Then $H = \Delta^\sigma L$.*

Note that the case $\text{rk} L = 1$ is excluded in Proposition 12 since the one-dimensional subgroup $\Delta S(U(1) \cdot U(1)) \subset SU(2) \times SU(2)$ acts polarly on $SU(2) = S^3$ with non-flat sections.

4. SUBACTIONS OF HERMANN ACTIONS

From now on assume that L is an exceptional simple compact Lie group E_6 , E_7 , E_8 , F_4 , or G_2 . Subgroups in groups of the form $\Delta^\sigma L$ have been treated in Section 3. By Lemma 3, we may assume H' is a closed connected subgroup of $H = H_1 \times H_2 \subset L \times L$, where $H_i \subset L$ are maximal closed connected subgroups.

We start with subactions of Hermann actions, i.e. the special case where both subgroups $H_1, H_2 \subset L$ are symmetric. All combinations where H_1 and H_2 are not

conjugate are given by Table 3, cf. [18]. (By $\mathrm{SO}'(2n)$ we denote the image of a half-spin representation of $\mathrm{Spin}(2n)$.)

Action	H_1	L	H_2	Coh.
E I-II	$\mathrm{Sp}(4)/\{\pm 1\}$	E_6	$\mathrm{SU}(6) \cdot \mathrm{Sp}(1)$	4
E I-III	$\mathrm{Sp}(4)/\{\pm 1\}$	E_6	$\mathrm{Spin}(10) \cdot \mathrm{U}(1)$	2
E I-IV	$\mathrm{Sp}(4)/\{\pm 1\}$	E_6	F_4	2
E II-III	$\mathrm{SU}(6) \cdot \mathrm{Sp}(1)$	E_6	$\mathrm{Spin}(10) \cdot \mathrm{U}(1)$	2
E II-IV	$\mathrm{SU}(6) \cdot \mathrm{Sp}(1)$	E_6	F_4	1
E III-IV	$\mathrm{Spin}(10) \cdot \mathrm{U}(1)$	E_6	F_4	1
E V-VI	$\mathrm{SU}(8)/\{\pm 1\}$	E_7	$\mathrm{SO}'(12) \cdot \mathrm{Sp}(1)$	4
E V-VII	$\mathrm{SU}(8)/\{\pm 1\}$	E_7	$E_6 \cdot \mathrm{U}(1)$	3
E VI-VII	$\mathrm{SO}'(12) \cdot \mathrm{Sp}(1)$	E_7	$E_6 \cdot \mathrm{U}(1)$	2
E VIII-IX	$\mathrm{SO}'(16)$	E_8	$E_7 \cdot \mathrm{Sp}(1)$	4
F I-II	$\mathrm{Sp}(3) \cdot \mathrm{Sp}(1)$	F_4	$\mathrm{Spin}(9)$	1

TABLE 3. Hermann actions on exceptional groups.

We will now restrict our attention to subactions of Hermann actions whose cohomogeneity is greater than one. The actions E II-IV, E III-IV F I-II and F II-II are of cohomogeneity one and their subactions will be treated below.

Theorem 13. *Let L be an exceptional simple compact Lie group and let H_1, H_2 be connected symmetric subgroups of L . Assume that the action of H on L is of cohomogeneity ≥ 2 . Let H' be a closed connected nontrivial subgroup of $H = H_1 \times H_2$. Then the action of H' on L is polar if and only if it is orbit equivalent to the H -action. Furthermore, there are orbit equivalent subactions only for the Hermann actions E II-III, E III-III and E VI-VII.*

Proof. First we will consider the special case where H_1 and H_2 are conjugate; we may assume $H_1 = H_2$. Then the isotropy subgroup of the H -action at the identity element $e \in L$ is $H \cap \Delta L = \Delta H_1 = \Delta H_2$ and the slice representation on $N_e(H \cdot e)$ is equivalent to the isotropy representation of the exceptional symmetric space $L/H_1 = L/H_2$. Assume $H' \subset H$ is a closed connected subgroup acting polarly on L . The isotropy subgroup of the H' -action at the identity element of L is $H'_e = H' \cap \Delta L$ and the slice representation of the H' -action contains the normal space $N_e(H \cdot e)$ to the H -orbit through e as a submodule. Since the slice representation of H_e on $N_e(H \cdot e)$ is irreducible of cohomogeneity ≥ 2 , it is polarity minimal by Proposition 9 (i). Thus the action of the group H'_e on $N_e(H \cdot e)$ is either orbit equivalent to the H_e -action or has finite orbits. In the latter case, there arises a contradiction with Theorem 6, since $N_e(H \cdot e)$ is an irreducible Lie triple system of higher rank in \mathfrak{p} .

Assume the H'_e -action on $N_e(H \cdot e)$ is orbit equivalent to the H_e -action. It then follows from the contents of Table 1 that H'_e contains the connected component of $H_e = \Delta H_1$, except possibly in case $L = E_6$, $H_1 = H_2 = \mathrm{Spin}(10) \cdot \mathrm{U}(1)$, where it follows only that H'_e contains a factor isomorphic to $\mathrm{Spin}(10)$.

Assume for the moment that H'_e contains all of ΔH_1 . Let $(h_1, h_2) \in H_1 \times H_2$ and consider the conjugate subgroup $(h_1, h_2) \cdot H' \cdot (h_1, h_2)^{-1} \subseteq H$, which also acts polarly on L . It follows from the above argument that also $(h_1, h_2) \cdot H' \cdot (h_1, h_2)^{-1}$ contains ΔH_1 . This shows that H' contains all conjugates of ΔH_1 in H , i.e. H' contains the smallest closed normal subgroup of H containing ΔH_1 . In case H is semisimple it follows that $H' = H$.

Now consider the case where H is not semisimple. There are two cases, corresponding to the symmetric spaces E III and E VII.

In case $L = E_6$ and $H_1 = \text{Spin}(10) \cdot \text{U}(1)$ it follows from the above argument that H' contains $\text{Spin}(10) \times \text{Spin}(10)$. Hence $H' \cong (\text{Spin}(10) \times \text{Spin}(10)) \cdot Q$, where Q is a closed subgroup of $\text{U}(1) \times \text{U}(1)$. Consider the H' -orbit $H' \cdot e$ through e ; it is a closed submanifold of codimension ≤ 1 in the orbit $H \cdot e$, which coincides with the subgroup $\text{Spin}(10) \cdot \text{U}(1) \subset E_6$ and it follows from Proposition 11 that $H' \cdot e = H \cdot e$. It follows that either $Q = \text{U}(1) \times \text{U}(1)$ or Q is any one-dimensional closed subgroup of $\text{U}(1) \times \text{U}(1)$ except the diagonal $\Delta \text{U}(1)$. In case $L = E_7$ and $H_1 = E_6 \cdot \text{U}(1)$, it follows from the above argument that H' contains $(E_6 \times E_6) \cdot \Delta \text{U}(1)$, however, this group acts non-polarly on E_7 by Proposition 11. It follows that $H' = H$ in this case.

Now consider the case where H_1 and H_2 are not conjugate. We assume that a

Action	Slice representation	Kernel
E I-II	$F_4/\text{Sp}(3) \cdot \text{Sp}(1)$	
E I-III	$\text{Sp}(4)/\text{Sp}(2) \cdot \text{Sp}(2)$	
E I-IV	$\text{SU}(6)/\text{Sp}(3)$	$\text{Sp}(1)$
E II-III	$\text{SU}(6)/\text{S}(\text{U}(2) \cdot \text{U}(4))$	$\text{Sp}(1)$
E II-IV	$\text{Sp}(4)/\text{Sp}(3) \cdot \text{Sp}(1)$	
E III-IV	$F_4/\text{Spin}(9)$	
E V-VI	$\text{SU}(8)/\text{S}(\text{U}(4) \cdot \text{U}(4))$	
E V-VII	$\text{SU}(8)/\text{Sp}(4)$	
E VI-VII	$\text{SU}(8)/\text{S}(\text{U}(2) \cdot \text{U}(6))$	
E VIII-IX	$\text{SO}(16)/\text{U}(8)$	
F I-II	$\text{Sp}(3)/\text{Sp}(2) \cdot \text{Sp}(1)$	$\text{Sp}(1)$

TABLE 4. Slice representations of Hermann actions on exceptional groups.

closed subgroup H' of $H = H_1 \times H_2$ acts polarly on L . In Table 4, an irreducible slice representation for each $H_1 \times H_2$ -action is given, cf. [19], Section 3.1.3. The entries of Table 4 are to be interpreted as follows: The action is given in the first column by the same notation as in Table 3; in the second column, a symmetric space whose isotropy representation is (on the Lie algebra level) equivalent to an effectivized slice representation is given; the third column states the local isomorphism type of the kernel of the slice representation (if nontrivial).

By an analogous argument as above it follows that the action of H'_e on the invariant subspace $N_e(H \cdot e)$ of the slice representation is orbit equivalent to the H_e -action on $N_e(H \cdot e)$, since we only consider actions of cohomogeneity ≥ 2 here.

We start with those Hermann actions where the slice representation restricted to the connected component of the isotropy group does not have orbit equivalent proper subgroups. Comparison of Tables 1 and 4 shows that this is the case for the actions E I-II, E I-III, E I-IV, E V-VI, E V-VII and E VIII-IX. We can read off the isomorphism type of the connected component of the group $H_e = \Delta(H_1 \cap H_2) \subset H$ from Table 4. As above, it follows that H' contains all conjugates of H_e in H . In case of the actions E I-II, E I-IV, E V-VI and E VIII-IX this shows that $H = H'$.

In case of the Hermann action E I-III, it follows that H' contains the subgroup $(\mathrm{SU}(4)/\{\pm 1\}) \times \mathrm{Spin}(10)$ of H . However, this group acts non-polarly on E_6 by Proposition 11 and Theorem 1 of [19], thus $H' = H$.

Similarly, for the action E V-VII, it follows from an analogous argument only that H' contains the subgroup $(\mathrm{SU}(8)/\{\pm 1\}) \times E_6$ of H and it follows again from Proposition 11 and Theorem 1 of [19] that $H' = H$.

For the action E II-III, it follows from the proof of Theorem 7.3 in [19] that H' contains the subgroup $(\mathrm{SU}(6) \cdot \mathrm{Sp}(1)) \times \mathrm{Spin}(10)$, whose action on E_6 is orbit equivalent to the H -action, cf. Theorem 2 of [19].

In case of the action E VI-VII we obtain that H'_e contains a subgroup $\mathrm{Sp}(1) \cdot \mathrm{SU}(6)$. This shows that H' contains a subgroup $(\mathrm{SO}'(12) \cdot \mathrm{Sp}(1)) \times E_6$, whose action on E_7 is orbit equivalent to the H -action, cf. Theorem 2 of [19]. \square \square

Theorem 14. *Let L be an exceptional simple compact Lie group and let H_1, H_2 be connected subgroups of L such that $H = H_1 \times H_2$ acts with cohomogeneity one on L . Let H' be a closed connected nontrivial subgroup of H . Then the action of H' on L is either non-polar or orbit equivalent to the H -action.*

The proof of Theorem 13 does not work for actions of cohomogeneity one, since their slice representations are also of cohomogeneity one and Proposition 9 (i) cannot be applied.

Proof. By the results of [18], there are the following cohomogeneity one Hermann actions on the simple compact exceptional Lie groups: E II-IV, E III-IV, F II-II and F I-II. We will now treat their subactions case by case. Here we use the classification of maximal connected subgroups in compact Lie groups, cf. [18], Section 2.1.

F II-II. Consider subgroups H' of $\mathrm{Spin}(9) \times \mathrm{Spin}(9)$ on F_4 . By Proposition 12, we may assume that H' is not contained in $\Delta\mathrm{Spin}(9)$. By Lemma 5 and Lemma 7 we may assume that $H' \subseteq H'_1 \times H'_2$, where H'_1 and H'_2 are one of the following

$$(7) \quad \mathrm{Spin}(8), \quad \mathrm{Spin}(7) \cdot \mathrm{SO}(2), \quad \mathrm{Spin}(6) \cdot \mathrm{Spin}(3), \quad \mathrm{Spin}(5) \cdot \mathrm{Spin}(4).$$

First assume $H'_1 = H'_2$. Since in this case the slice representation at the identity element of F_4 is equivalent to the isotropy representation of the homogeneous space F_4/H'_1 , it follows from Theorem 2 of [20] that the H' -action is non-polar.

The remaining actions not excluded by these arguments can be seen to have non-polar slice representations.

F II-I. Assume H' is closed connected subgroup of $H = H_1 \times H_2 = \mathrm{Spin}(9) \times (\mathrm{Sp}(3) \cdot \mathrm{Sp}(1))$ acting polarly on F_4 . By Lemma 5 we know that H' does not contain $H_1 \times \{e\}$ or $\{e\} \times H_2$. The maximal dimension of a proper closed subgroup in H_1 or H_2 is 28 and 22, respectively. Hence it follows from Lemma 7 that H' is

contained in a subgroup $H'_1 \times H'_2$ where H'_1 is a maximal connected subgroup of $\text{Spin}(9)$ of dimension ≥ 18 and H'_2 is a maximal connected subgroup of $\text{Sp}(3) \cdot \text{Sp}(1)$ of dimension ≥ 12 . The maximal connected subgroups in H_1 of dimension ≥ 18 are:

$$(8) \quad \text{Spin}(8), \quad \text{Spin}(7) \cdot \text{SO}(2), \quad \text{Spin}(6) \cdot \text{Spin}(3).$$

The maximal connected subgroups of H_2 of dimension ≥ 12 are:

$$(9) \quad \text{Sp}(3) \cdot \text{U}(1), \quad (\text{Sp}(2) \cdot \text{Sp}(1)) \cdot \text{Sp}(1), \quad \text{U}(3) \cdot \text{Sp}(1).$$

We first consider the action of $\text{Spin}(8) \times (\text{Sp}(3) \cdot \text{Sp}(1))$ on F_4 ; it has an isotropy subgroup whose connected component is isomorphic to $\text{Sp}(2) \cdot \text{Sp}(1)$ and whose slice representation is $(\mathbb{H}^2 \otimes_{\mathbb{H}} \mathbb{H}^1) \oplus \mathbb{R}^5$, see [19], Section 12, p. 479. This representation is non-polar [1] and it can be verified that it is polarity minimal by looking at the closed subgroups of $\text{Sp}(2) \cdot \text{Sp}(1)$. Thus the action on F_4 is polarity minimal by Lemma 10 (ii).

Now consider the action of $(\text{Spin}(7) \cdot \text{SO}(2)) \times (\text{Sp}(3) \cdot \text{U}(1))$ on F_4 . An explicit calculation shows that there is an isotropy subgroup locally isomorphic to $\text{Spin}(5) \cdot \text{U}(1) \cdot \text{U}(1)$ whose non-polar slice representation splits as a direct sum $\mathbb{R}^8 \oplus (\mathbb{R}^5 \otimes \mathbb{R}^2) \oplus \mathbb{R}^2$. This representation is non-polar [1]. Using Table 1 and Proposition 9 (i) we see that an 18-dimensional submodule is polarity minimal.

The action of $(\text{Spin}(6) \cdot \text{Spin}(3)) \times (\text{Sp}(3) \cdot \text{U}(1))$ has a non-polar slice representation. Its subactions are ruled out by a dimension count. All other combinations of the groups in (8) and (9) result in actions which are non-polar by Lemma 7.

E II-IV. Assume H' is a closed subgroup of $H = (\text{SU}(6) \cdot \text{Sp}(1)) \times F_4$ acting polarly on E_6 . By Lemma 7, we have $\dim H' \geq 60$. By Lemma 5 it follows that H' is contained in a subgroup $H'_1 \times H'_2$, where $H'_1 \subset \text{SU}(6) \cdot \text{Sp}(1)$ and $H'_2 \subset F_4$ are maximal connected subgroups. Since a maximal subgroup of maximal dimension in $\text{SU}(6) \cdot \text{Sp}(1)$ is $\text{SU}(6) \cdot \text{U}(1)$, we may assume that $H'_2 = \text{Spin}(9)$ or $\text{Sp}(3) \cdot \text{Sp}(1)$, cf. Table 6. The remaining possibilities for $H'_1 \times H'_2$ are

$$\begin{aligned} & (\text{S}(\text{U}(5) \cdot \text{U}(1)) \cdot \text{Sp}(1)) \times \text{Spin}(9), \quad (\text{Sp}(3) \cdot \text{Sp}(1)) \times \text{Spin}(9), \\ & (\text{SU}(6) \cdot \text{U}(1)) \times \text{Spin}(9), \quad (\text{SU}(6) \cdot \text{U}(1)) \times (\text{Sp}(3) \cdot \text{Sp}(1)). \end{aligned}$$

We first consider the cases where $H'_2 = \text{Spin}(9)$. From Table 4 we see that there is an isotropy group of the action E II-IV whose connected component is $\text{Sp}(3) \cdot \text{Sp}(1)$. First we determine the intersection of this group with $\text{Spin}(9)$, from the last entry of Table 4 we see that, possibly after conjugation, $\text{Sp}(3) \cdot \text{Sp}(1) \cap \text{Spin}(9) = \text{Sp}(2) \cdot \text{Sp}(1) \cdot \text{Sp}(1)$. The slice representation of this isotropy group contains two submodules equivalent to $\mathbb{H}^2 \otimes_{\mathbb{H}} \mathbb{H}^1$ and is thus non-polar by [18], Lemma 2.9 and polarity minimal by Proposition 9 (ii). Thus the $H'_1 \times H'_2$ -action on E_6 is polarity minimal by Theorem 6.

In case of the last group we see from [7], Table 25, p. 200, that there is only one conjugacy class of a subgroup of type C_3 in E_6 and this has a 3-dimensional centralizer. It follows that there is an isotropy subgroup $\text{Sp}(3) \cdot \text{U}(1)$, whose 40-dimensional slice representation is the restriction of the isotropy representation of $E_6/(\text{SU}(6) \cdot \text{Sp}(1))$. This representation is non-polar by Proposition 9 (i) and Table 1. Subactions are non-polar by Lemma 7.

E III-IV. By Lemma 5, we may assume that any compact subgroup of $(\text{Spin}(10) \cdot \text{U}(1)) \times F_4$ acting polarly with non-flat sections on E_6 is contained in $H'_1 \times H'_2$ where H'_1 is a maximal connected subgroup of $\text{Spin}(10) \cdot \text{U}(1)$ and H'_2 is a maximal

connected subgroup of F_4 . It follows from Lemma 7 that we may assume H'_1 is one of

$$\begin{aligned} &\text{Spin}(10), \quad \text{Spin}(9) \cdot \text{U}(1), \quad \text{Spin}(8) \cdot \text{SO}(2) \cdot \text{U}(1), \\ &\text{Spin}(7) \cdot \text{Spin}(3) \cdot \text{U}(1), \quad \text{U}(5) \cdot \text{U}(1) \end{aligned}$$

and that H'_2 is one of

$$\text{Spin}(9), \quad \text{Sp}(3) \cdot \text{Sp}(1), \quad \text{G}_2^1 \cdot \text{A}_1^8, \quad \text{SU}(3) \cdot \text{SU}(3),$$

cf. Table 6. First assume $H'_2 = \text{Spin}(9)$. The group $\text{Spin}(9)$ also occurs as an isotropy group of the F_4 -action on $E_6/(\text{Spin}(10) \cdot \text{U}(1))$, cf. Table 4, and we see that any action of a group $H'_1 \times \text{Spin}(9)$ has a slice representation with two equivalent submodules. Such a representation is non polar by [18], Lemma 2.9 and polarity minimal by Lemma 9 (ii). Since the sum of these two submodules is 32-dimensional, the $H'_1 \times H'_2$ -action is non-polar and polarity minimal, cf. Lemma 10. This argument shows in particular that we may assume $\dim H'_2 \leq 24$. Then the remaining possibilities for $H'_1 \times H'_2$ are

$$\begin{aligned} &\text{Spin}(10) \times (\text{Sp}(3) \cdot \text{Sp}(1)), \quad (\text{Spin}(9) \cdot \text{U}(1)) \times (\text{Sp}(3) \cdot \text{Sp}(1)), \\ &\text{Spin}(9) \times (\text{Sp}(3) \cdot \text{Sp}(1)), \quad \text{Spin}(10) \times (\text{G}_2^1 \cdot \text{A}_1^8) \end{aligned}$$

In case of the first three actions, an isotropy group is $\text{Sp}(2) \cdot \text{Sp}(1) \cdot \text{Sp}(1)$. We can read off from Table 4 that in both cases the $\text{Sp}(2)$ -factor acts nontrivially on at least two factors of the slice representation, which is hence non-polar [1]. Subactions can be excluded by Lemma 7.

It remains to study the action of $\text{Spin}(10) \times (\text{G}_2^1 \cdot \text{A}_1^8)$ on E_6 ; however, by Table 39 of [7], the subgroup $\text{G}_2^1 \cdot \text{A}_1^8$ of F_4 is contained in the subgroup $\text{G}_2^1 \cdot \text{A}_2^{2''}$ of E_6 . Hence this action is a subaction of (6)–(10) which will be shown to be non-polar and polarity minimal in Section 7. \square \square

5. ACTIONS ON G_2

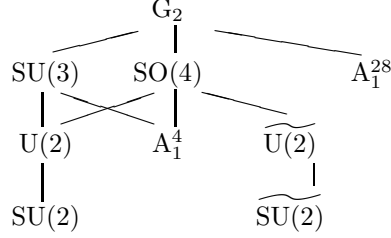
In this section, we will study those isometric actions on G_2 which are neither subactions of the ΔG_2 -action nor of the $\text{SO}(4) \times \text{SO}(4)$ -action.

1. Subgroups of G_2 .

Proposition 15. *All conjugacy classes of closed connected nonabelian subgroups of G_2 and their inclusion relations are given by Table 5.*

Remark. In Table 5, a tilde is used to distinguish between nonconjugate isomorphic subgroups; e.g. the groups denoted by $\text{SU}(2)$ and $\widetilde{\text{SU}(2)}$ correspond to the subgroups denoted by A_1 and \tilde{A}_1 , respectively, in [7]. By the upper indices, the Dynkin index of subgroups is given. Two subgroups H_1, H_2 are connected by a line if and only if there is an element $g \in G_2$ such that an inclusion relation holds between H_1 and $g H_2 g^{-1}$.

Proof. It is straightforward to prove the proposition using the results of [7]. The conjugacy classes of three dimensional (hence simple) connected subgroups of G_2 are given in Table 16 of [7]; there are four classes, distinguished by their Dynkin indices, which are 1, 3, 4 and 28. These subgroups are denoted in [7] by A_1, \tilde{A}_1, A_1^4 and A_1^{28} , respectively. There are two conjugacy classes of maximal regular connected subgroups, $\text{SU}(3) = A_2^1$ and $\text{SO}(4) = A_1 \cdot \tilde{A}_1$; moreover, there is only one

TABLE 5. Conjugacy classes of nonabelian connected subgroups in G_2 .

conjugacy class of connected subgroups not contained in a proper regular subgroup and this is A_1^{28} . The maximal connected subgroups of $SO(4) = A_1 \cdot \tilde{A}_1$ are $SO(3)$ and the two subgroups which we denote by $U(2)$ and $\widetilde{U(2)}$, containing the two simple factors A_1 and \tilde{A}_1 of $SO(4)$, which are not conjugate in G_2 , since they have different Dynkin indices. It follows from Table 16 of [7] that $SO(3)$ corresponds to the group A_1^4 . The maximal connected subgroups of $SU(3)$ are $S(U(2) \cdot U(1))$ and $SO(3)$. Since the first group has Dynkin index one as a subgroup of G_2 , it follows that it corresponds to the subgroup denoted by $U(2)$ in Table 5, the second group obviously corresponds to A_1^4 . \square

It follows from Table 5 that all connected proper subgroups of G_2 except $SU(3)$ and A_1^{28} are contained in $SO(4)$ after conjugation with a suitable element from G_2 . Since the $SO(4) \times SO(4)$ -action and the ΔG_2 -action were already shown to be polarity minimal, it suffices to consider the actions of subgroups $H \subseteq H_1 \times H_2$ where at least one of the closed connected subgroups $H_1, H_2 \subsetneq G_2$ is conjugate to either $SU(3)$ or A_1^{28} . Let $\pi_i: (g_1, g_2) \mapsto g_i$ for $i = 1, 2$ be the canonical projections $G_2 \times G_2 \rightarrow G_2$.

Let us first consider the case where at least one of the factors $\pi_1(H)$ or $\pi_2(H)$ is conjugate to $SU(3)$. We may assume w.l.o.g. $\pi_2(H) = SU(3)$. Now if $\pi_1(H)$ is conjugate to $SU(3)$, too, then H is conjugate to either $SU(3) \times SU(3)$ or $\Delta SU(3)$; in the first case, the action is a well known cohomogeneity one action, in the latter case, the H -action on G_2 is non-polar by Proposition 12, since it is not orbit equivalent to the ΔG_2 -action. If $\pi_1(H)$ is not conjugate to $SU(3)$, then it follows that H is of the form $H_1 \times H_2$, where $H_1 = \ker \pi_2|_H$ and $H_2 = \ker \pi_1|_H \cong SU(3)$. We will consider this case in Subsection 2. It remains the case where one of the factors $\pi_1(H)$ or $\pi_2(H)$ is conjugate to A_1^{28} , which we will treat in Subsection 3.

2. Actions of $H_1 \times SU(3)$ on G_2 .

Lemma 16. *Let $H_1 \subset G_2$ be a closed subgroup. If the cohomogeneity of the action of $H_1 \times SU(3)$ on G_2 is greater than one then the action is not polar.*

Proof. Assume the action of $H_1 \times H_2$ on G_2 is polar, where $H_2 = SU(3)$. Let \mathfrak{h}_i be the Lie algebra of H_i and let \mathfrak{m}_i be the orthogonal complement of \mathfrak{h}_i in \mathfrak{g}_2 for $i = 1, 2$. We may assume that the identity element $e \in G_2$ lies in a principal orbit. Using Proposition 4 (ii), it follows in particular that $p([\nu, \nu]) = 0$, where $\nu = \mathfrak{m}_1 \cap \mathfrak{m}_2 = N_e(H_1 \times H_2) \cdot e$. and where p denotes the orthogonal projection

$\mathfrak{g}_2 \rightarrow \mathfrak{h}_2$. We want to study the \mathbb{R} -bilinear map $\beta: \mathfrak{m}_2 \times \mathfrak{m}_2 \rightarrow \mathfrak{h}_2$ given by $(x, y) \mapsto p([x, y])$.

To describe this map explicitly, consider the antisymmetric bilinear map $F: \mathbb{C}^3 \times \mathbb{C}^3 \rightarrow \mathfrak{u}(3)$, $(x, y) \mapsto x\bar{y}^t - y\bar{x}^t$, which maps a pair of vectors in \mathbb{C}^3 to a skew-hermitian matrix. Define $F_0: \mathbb{C}^3 \times \mathbb{C}^3 \rightarrow \mathfrak{su}(3)$ by $F_0(x, y) := F(x, y) - \frac{1}{3} \text{tr}(F(x, y)) I$, where I denotes the 3×3 identity matrix. Let $A \in \text{SU}(3)$; then $F_0(Ax, Ay) = A \cdot F(x, y) \cdot A^{-1}$, i.e. F_0 defines a non-zero, hence surjective, $\text{SU}(3)$ -equivariant map $\Lambda^2 \mathbb{R}^6 \rightarrow \mathfrak{su}(3)$.

The adjoint representation of G_2 restricted to H_2 leaves \mathfrak{m}_2 invariant and the action of H_2 on \mathfrak{m}_2 is equivalent to the natural $\text{SU}(3)$ -representation on $\mathbb{C}^3 = \mathbb{R}^6$. We may thus identify \mathfrak{m}_2 with \mathbb{C}^3 by an $\text{SU}(3)$ -equivariant \mathbb{R} -linear isomorphism. Since $\Lambda^2 \mathbb{R}^6$, considered as an $\text{SU}(3)$ -module, contains only one irreducible summand equivalent to the adjoint representation of $\text{SU}(3)$, it follows from Schur's Lemma that β and F_0 agree up to an equivariant isomorphism (β is non-zero since otherwise \mathfrak{m}_2 would be an ideal of \mathfrak{g}_2).

Now assume x, y are two non-zero elements in $\nu \subseteq \mathfrak{m}_2 = \mathbb{C}^3$ such that $F_0(x, y) = 0$. We will show that x and y are linearly dependent over \mathbb{R} . Since F_0 is \mathbb{R} -bilinear and $\text{SU}(3)$ acts transitively on the unit sphere in \mathbb{R}^6 , we may assume that $x = (1, 0, 0)^t \in \mathbb{C}^3$. Then we have

$$F_0(x, y) = \begin{pmatrix} \frac{2}{3}(\bar{y}_1 - y_1) & \bar{y}_2 & \bar{y}_3 \\ -y_2 & -\frac{1}{3}(\bar{y}_1 - y_1) & 0 \\ -y_3 & 0 & -\frac{1}{3}(\bar{y}_1 - y_1) \end{pmatrix},$$

where $y = (y_1, y_2, y_3)^t$; this shows that $F_0(x, y) = 0$ if and only if $x = \alpha y$ for some $\alpha \in \mathbb{R}$. Hence ν is at most one-dimensional and the cohomogeneity of the action is at most one. □ □

Assume now $H = H_1 \times \text{SU}(3)$ acts polarly, hence with cohomogeneity one, on G_2 . It follows that $\dim(H_1) \geq 5$. A glance at Table 5 now shows that $H_1 = \text{SU}(3)$ or $\text{SO}(4)$. The actions of these groups are of cohomogeneity one [18].

3. Subactions of $H_1 \times A_1^{28}$ on G_2 . Now consider the case where one of the factors $\pi_1(H)$ or $\pi_2(H)$ is conjugate to A_1^{28} . Assume H acts polarly on G_2 . By Lemma 7, we have $\dim(H) \geq 8$ and the only possibilities for H are $A_1^{28} \times \text{SU}(3)$ and $A_1^{28} \times \text{SO}(4)$. The first action is non-polar by Lemma 16. It remains to study the action of $H = \text{SO}(4) \times A_1^{28}$ on G_2 . This action is non-polar by Lemma 5 and the results of [18]. Since any proper closed subgroup of H is of dimension ≤ 7 , the action is polarity minimal by Lemma 7.

6. ACTIONS ON F_4

We will now study actions on F_4 which are neither subactions of σ -actions nor of Hermann actions. By Lemma 3 we may assume that the group acting polarly is contained in $H_1 \times H_2$, where H_i are maximal connected subgroups. By Lemma 8 it follows that $\dim H_i \geq 12$. The conjugacy classes of all such subgroups of F_4 are given by Table 6, cf. [7]. In the case of a symmetric subgroup the type of the symmetric space is given in the last column. Of course we do not need to consider groups $H_1 \times H_2$ where H_1 and H_2 are both symmetric subgroups, since they have already been considered in Section 4. We will follow the same procedure in Sections 7 and 8 for actions on the groups E_6 and E_7 .

No.	Subgroup	Dimension	Type
(1)	$\text{Spin}(9)$	36	F II
(2)	$\text{Sp}(3) \cdot \text{Sp}(1)$	24	F I
(3)	$G_2^1 \cdot A_1^8$	17	
(4)	$\text{SU}(3) \cdot \text{SU}(3)$	16	

TABLE 6. Maximal connected subgroups of F_4 of dimension ≥ 12 .

(1)–(3). We determine a slice representation of the action of $H = H_1 \times H_2$, where $H_1 = G_2^1 \cdot A_1^8$ and $H_2 = \text{Spin}(9)$, on F_4 . Consider the subgroup $G_2 \subset \text{Spin}(7) \subset \text{Spin}(9)$. By [7], Table 25, p. 199, there is only one conjugacy class of subalgebras isomorphic to G_2 and it follows that there is an isotropy subgroup containing the G_2 -factor of the group **(3)**. By Table 25 in [7], the dimension of the normal space is a multiple of 7. A dimension count shows that such an isotropy group is of dimension 15 and thus its Lie algebra is isomorphic to $\mathfrak{g}_2 \oplus \mathbb{R}$. From the fact that $F_4/G_2^1 \cdot A_1^8$ is a strongly isotropy irreducible homogeneous space [27], one can deduce that the 14-dimensional slice representation is orbit equivalent to the action of $G_2 \times \text{SO}(2)$ on $\mathbb{R}^7 \otimes \mathbb{R}^2$ given by the tensor product of the 7-dimensional irreducible G_2 -representation and a non-trivial 2-dimensional real representation of $\text{SO}(2)$. Thus the action is of cohomogeneity two (see Table 1) and non-polar by Lemma 5. By Table 1 and since the 14-dimensional slice representation is polarity minimal, it follows that any closed subgroup H' of H acting polarly on F_4 must contain ΔG_2^1 . An argument similar as in the proof of Theorem 13 now shows that H' contains $G_2^1 \times \text{Spin}(9)$. Hence the H' -action is non-polar by Lemma 5.

(1)–(4). The action of $H_2 = \text{SU}(3) \cdot \text{SU}(3)$ on the Cayley plane $F_4/\text{Spin}(9)$ is polar of cohomogeneity two, see [25], hence with non-flat sections. It follows from Lemma 5 that the action of $\text{Spin}(9) \times H_2'$ on F_4 is non-polar for all closed subgroups $H_2' \subseteq \text{SU}(3) \cdot \text{SU}(3)$. Assume a group H' acting polarly on F_4 is contained in $H_1' \times (\text{SU}(3) \cdot \text{SU}(3))$ where H_1' is a maximal connected subgroup of $\text{Spin}(9)$. By Lemma 7 it follows that $H_1' = \text{Spin}(8)$. Consider the action of $\text{Spin}(8) \times (\text{SU}(3) \cdot \text{SU}(3))$ on F_4 . By a calculation as in [19], Remark 10.1, one finds an isotropy subgroup $\text{SU}(3) \cdot T^2 = \text{U}(3) \cdot \text{SO}(2)$. The 18-dimensional slice representation, when restricted to $\text{SU}(3)$, splits into three times the standard representation on \mathbb{R}^6 . This representation is non-polar and polarity minimal [1], [5].

(2)–(3) and (2)–(4). Let $H_1 = \text{Sp}(3) \cdot \text{Sp}(1)$ and let $H_2 = G_2^1 \cdot A_1^8$ or $\text{SU}(3) \cdot \text{SU}(3)$. By the results of [19] and Lemma 5, the action of $H = H_1 \times H_2$ on F_4 is non-polar. Since any proper closed subgroup of H is of dimension ≤ 39 , it follows from Lemma 7 that no closed connected nontrivial subgroup of H acts polarly on F_4 .

7. ACTIONS ON E_6

We follow the same procedure as in Section 6. The maximal connected subgroups of dimension greater or equal to 15 are given in Table 7.

(5)–(9) and (5)–(10). For both actions, a slice representation is computed in [19], Subsection 10.1. It is shown there that these representations are non-polar and polarity minimal. Since these slice representations are of dimension 36 and 21, respectively, it follows from Lemma 10 (ii) that both actions on E_6 are non-polar and polarity minimal.

No.	Subgroup	Dimension	Type
(5)	F_4	52	E IV
(6)	$\text{Spin}(10) \cdot \text{U}(1)$	46	E III
(7)	$\text{SU}(6) \cdot \text{Sp}(1)$	38	E II
(8)	$\text{Sp}(4)/\{\pm 1\}$	36	E I
(9)	$\text{SU}(3) \cdot \text{SU}(3) \cdot \text{SU}(3)$	24	
(10)	$G_2^1 \cdot A_2^{2''}$	22	

TABLE 7. Maximal connected subgroups of E_6 of dimension ≥ 15 .

(6)–(9). See Section 10.

(6)–(10). Let $H_1 = \text{Spin}(10) \cdot \text{U}(1)$, $H_2 = G_2^1 \cdot A_2^{2''}$. Using Table 25 of [7], pp. 200 and 203, it follows that G_2^1 is contained (after conjugation) in H_1 . Since the isotropy representation of the strongly isotropy irreducible space $E_6/(G_2^1 \cdot A_2^{2''})$ is equivalent to the tensor product of the adjoint representation of $\text{SU}(3)$ and the 7-dimensional irreducible representation of G_2 , it follows that the dimension of the slice representation of $H_1 \cap H_2$ is a multiple of 7. A dimension count now shows that the isotropy group $H_1 \cap H_2$ is locally isomorphic to $G_2 \cdot \text{S}(\text{U}(2) \cdot \text{U}(1))$ and the 28-dimensional slice representation is non-polar [5], [1] and polarity minimal. Thus the $H_1 \times H_2$ -action is non-polar and polarity minimal by Proposition 10 (ii).

(7)–(10) and (8)–(9). The actions of the groups $H = (\text{SU}(6) \cdot \text{Sp}(1)) \times (G_2^1 \cdot A_2^{2''})$ and $(\text{Sp}(4)/\{\pm 1\}) \times (\text{SU}(3) \cdot \text{SU}(3) \cdot \text{SU}(3))$ are non-polar by Lemma 5 and the results of [18]. Since both groups are 60-dimensional, it follows from Lemma 7 that no closed connected nontrivial subgroup of these groups acts polarly on E_6 .

(7)–(9). See Section 10.

8. ACTIONS ON E_7

The maximal connected subgroups of E_7 of dimension ≥ 34 are given in Table 8, cf. [7]. The groups (11), (12) and (13) are the symmetric subgroups of E_7 .

No.	Subgroup	Dimension	Type
(11)	$E_6 \cdot \text{U}(1)$	79	E VII
(12)	$\text{SO}'(12) \cdot \text{Sp}(1)$	69	E VI
(13)	$\text{SU}(8)/\{\pm 1\}$	63	E V
(14)	$F_4^1 \cdot A_1^{3''}$	55	
(15)	$\text{SU}(6) \cdot \text{SU}(3)$	43	
(16)	$G_2^1 \cdot C_3^{1''}$	35	

TABLE 8. Maximal connected subgroups of E_7 of dimension ≥ 34 .

(11)–(14) and (12)–(14). Let $H_1 = F_4^1 \cdot A_1^{3''}$. For the cases $H_2 = E_6 \cdot \text{U}(1)$ or $H_2 = \text{SO}'(12) \cdot \text{Sp}(1)$, the argument given in [19], Section 10.2 shows that the H -action on L is non-polar and polarity minimal by Lemma 10 (ii).

(11)–(15) and (12)–(15). See Section 10.

(11)–(16). It follows from [7], Table 25, that the group G_2^1 is contained (after conjugation) in $E_6 \subset E_7$ and that the dimension of a slice representation of an isotropy subgroup containing G_2^1 is a multiple of 7. Now consider the subgroup $G_2^1 \cdot A_2^{2''}$ of E_6 . It can be read off from Table 25 in [7] that this group is contained in the subgroup $G_2^1 \cdot C_3^{1''}$ of E_7 . A dimension count shows that an isotropy group containing $G_2^1 \cdot C_3^{1''}$ must be locally isomorphic to $G_2 \cdot U(3)$. The corresponding slice representation is equivalent to the 42-dimensional real tensor product of the 7-dimensional G_2 -representation and the real 6-dimensional standard $SU(3)$ -representation. This slice representation is irreducible and non-polar [5], hence polarity minimal by Proposition 9. We conclude that the action (11)–(16) is non-polar and polarity minimal by Lemma 10 (ii).

(13)–(14). The action of $(SU(8)/\{\pm 1\}) \times H_2'$ on E_7 , where H_2' is a closed subgroup of $F_4^1 \cdot A_1^{3''}$, is non-polar by Lemma 5. The dimension of any other proper closed subgroup in $(SU(8)/\{\pm 1\}) \times F_4^1 \cdot A_1^{3''}$ is less than 105.

9. ACTIONS ON E_8

Since all closed connected subgroups of E_8 of dimension ≥ 90 are symmetric [7], it follows from Lemma 8 and Theorem 13 that any polar action on E_8 is hyperpolar or has finite orbits.

10. REGULAR SUBGROUPS OF THE ISOMETRY GROUP

Action	Isotropy subgroup	Slice representation
(6)–(9)		$\overset{1}{\bullet} \text{---} \bullet \quad \overset{1}{\bullet} \quad \bullet \oplus \overset{1}{\bullet} \text{---} \bullet \quad \bullet \quad \overset{1}{\bullet}$
(7)–(9)		$\overset{1}{\bullet} \text{---} \bullet \quad \bullet \text{---} \overset{1}{\bullet} \quad \overset{1}{\bullet}$
(11)–(15)		$\overset{1}{\bullet} \text{---} \bullet \text{---} \bullet \text{---} \bullet \quad \overset{1}{\bullet} \oplus \bullet \text{---} \bullet \text{---} \overset{1}{\bullet} \text{---} \bullet \quad \bullet$
(12)–(15)		$\bullet \text{---} \overset{1}{\bullet} \text{---} \bullet \text{---} \bullet \text{---} \bullet \quad \overset{1}{\bullet}$

TABLE 9. Isotropy groups and slice representations of certain regular subgroups.

For the special case where H_1, H_2 are maximal regular subgroups of the simple compact Lie group G , one can determine a slice representation of the $H_1 \times H_2$ -action on G by the method described in Remark 10.1 of [19], see also §3 of [22], Theorem 16. We will apply this method now to certain actions on E_6 and E_7 .

In the middle column of Table 9, the extended Dynkin diagram of G is given for each action under consideration. We assume that the intersection of H_1 and H_2 contains a fixed maximal torus T of G . Then the root systems of H_1 and H_2 with respect to T are subsets S_1, S_2 of the root system R of G . Simple roots of the root systems of H_1 or H_2 are shown in Table 9 by black nodes \bullet or circles \circ , respectively. The intersection $S_1 \cap S_2$ of both root system is the root system of the isotropy group $H_1 \cap H_2$, its simple roots are shown as \odot . The roots in $R \setminus (S_1 \cup S_2)$ are exactly the weights of the slice representation of $H_1 \cap H_2$. In the third column of Table 9, highest weights of the irreducible submodules of the slice representation, restricted to the semisimple part of $H_1 \cap H_2$, and viewed as complex representations, are given. The slice representations are of dimension 24, 36, 40 and 60, respectively; they are non-polar [5], [1] and polarity minimal by Proposition 9. Thus all four actions are non-polar and polarity minimal by Lemma 10 (ii).

11. PRINCIPAL ISOTROPY ALGEBRAS OF ACTIONS ON G_2

In this section, we determine all isometric actions on the compact exceptional Lie group G_2 which have principal isotropy subgroups of positive dimension.

Theorem 17. *Let $H \subseteq G_2 \times G_2$ be a closed connected subgroup acting nontransitively on G_2 . Then the principal isotropy groups of the H -action on G_2 are finite except if H is conjugate to ΔG_2 , $SU(3) \times SU(3)$, $SU(3) \times SO(4)$, or $SO(4) \times SU(3)$.*

In particular, it follows from Theorem 17 and Table 5 that for all integers $0 \leq d \leq 14$ there is a closed subgroup $H \subset G_2 \times G_2$ such that the H -action on G_2 has principal orbits of dimension d .

Proof. We will use the same kind of recursion procedure as in the proof of Theorem 18 to classify all closed subgroups $H \subset G_2 \times G_2$ such that H acts nontransitively on G_2 with principal isotropy groups of positive dimension. By Lemma 3, we may assume that either H is contained in ΔG_2 or that H is contained in a group of the form $H_1 \times H_2$ where $H_i \subset G_2$ are maximal connected subgroups, cf. Table 5.

Assume first $H \subseteq \Delta G_2$. For the actions of such groups the identity element $e \in G_2$ is a fixed point and we may consider the slice representations of these actions at e , which are equivalent to the adjoint representation of G_2 restricted to H . The action of ΔG_2 on G_2 is the adjoint action whose principal isotropy groups are the maximal tori of G_2 . Now consider the maximal connected subgroups $H' = \Delta SU(3)$, $\Delta SO(4)$ and ΔA_1^{28} of $\Delta G_2 \cong G_2$, cf. Table 5. In all three cases the adjoint representation of G_2 restricted to H' is equivalent to the adjoint representation of H' plus the isotropy representation of the strongly isotropy irreducible homogeneous space G_2/H' [27]. We see from [15] that these representations have finite principal isotropy groups. Hence all subactions of these action also have finite principal isotropy groups.

Now consider H contained in groups of the form $H' = H_1 \times H_2$, where $H_i \in \{SU(3), SO(4), A_1^{28}\}$. The action of $SO(4) \times SO(4)$ on G_2 has finite principal isotropy groups; therefore, in order to find all closed subgroups of $H' = H_1 \times H_2$ acting with principal isotropy groups of positive dimension, we may assume that

$H_1 = \mathrm{SU}(3)$ or $H_1 = \mathrm{A}_1^{28}$ and that H contains the H_1 -factor of H' . We will treat the remaining possibilities according to Table 5.

Assume $H_1 = \mathrm{SU}(3)$. If $H_2 = \mathrm{SU}(3)$, then the H' -action on G_2 is of cohomogeneity one with principal isotropy group $\Delta\mathrm{SU}(2)$, see [18]. If $H = \mathrm{SU}(3) \times \mathrm{U}(2)$, then an isotropy group of the H -action on G_2 is $\Delta\mathrm{U}(2)$ whose slice representation is equivalent to the standard representation of $\mathrm{SU}(3)$ restricted to $\mathrm{S}(\mathrm{U}(2) \cdot \mathrm{U}(1))$, which has trivial principal isotropy. If $H = \mathrm{SU}(3) \times \mathrm{A}_1^4$, then the principal isotropy is also trivial since a slice representation is equivalent to the standard representation of $\mathrm{SO}(3)$ on \mathbb{C}^3 . The action of $H = \mathrm{SU}(3) \times \mathrm{SO}(4)$ is of cohomogeneity one [18]. Let us show that the action of $H = \mathrm{SU}(3) \times \widetilde{\mathrm{U}(2)}$ on G_2 is of cohomogeneity two. The 7-dimensional irreducible representation of G_2 splits as $\mathbb{R}^3 \oplus \mathbb{R}^4$ when restricted to $\widetilde{\mathrm{U}(2)}$, where $\widetilde{\mathrm{U}(2)}$ acts by the standard $\mathrm{U}(2)$ -representation on \mathbb{R}^4 and by the adjoint representation of $\mathrm{SU}(2)$ on \mathbb{R}^3 . This representation, and hence the action of $\widetilde{\mathrm{U}(2)}$ on $\mathrm{S}^6 = G_2/\mathrm{SU}(3)$ has finite principal isotropy groups. The action of $\mathrm{SO}(3)$ on S^6 given by the 7-dimensional irreducible representation of $\mathrm{SO}(3)$ has trivial principal isotropy groups [15], hence it follows that the principal isotropy groups of the $\mathrm{SU}(3) \times \mathrm{A}_1^{28}$ -action on G_2 are trivial as well.

Now assume $H_1 = \mathrm{A}_1^{28}$. Consider the action of $\mathrm{A}_1^{28} \times \mathrm{A}_1^{28}$ on G_2 . Since by [27] one slice representation is equivalent to the 11-dimensional irreducible representation of $\mathrm{SO}(3)$, it follows that the action has trivial principal isotropy groups [15].

Assume the action of $H' = \mathrm{A}_1^{28} \times \mathrm{SO}(4)$ on G_2 has principal isotropy groups of positive dimension. Then it follows that the action is of cohomogeneity at least six. We may assume that the identity element $e \in G_2$ lies in a principal orbit. Then there is some non-zero element $X \in \mathfrak{h}' \cap \Delta\mathfrak{g}_2$ acting trivially on the normal space $N_e(H' \cdot e) \subset \mathfrak{g}_2$. But this contradicts the fact that the centralizer of any non-zero element in \mathfrak{g}_2 is at most 4-dimensional. \square \square

12. ACTIONS OF COHOMOGENEITY ONE OR TWO

In this section we classify all isometric actions of cohomogeneity less or equal to two on the exceptional simple compact Lie groups $L = G_2, F_4, E_6, E_7$ and E_8 . By the results of [21], there are no nontrivial transitive actions on the exceptional groups. In [18], a classification of cohomogeneity one actions on simple compact Lie groups was obtained. However, these actions were only classified up to orbit equivalence there and it remains to determine orbit equivalent subactions.

Obviously, we only need to consider such closed subgroups H of $L \times L$ where $\dim H \geq \dim L - 2$. Since the lower bound on the dimension of groups acting polarly on L given in Lemmata 7 and 8 is in all cases a weaker condition, all candidates for groups acting with cohomogeneity one or two have already appeared in the proof of Theorem 1. Indeed, in order to prove the classification theorems 18 and 19 below, we will proceed in the same order as in the proof of Theorem 1, i.e. we first consider subgroups of ΔL , then we study subactions of Hermann actions and finally we consider all remaining subgroups of $L \times L$ which are of sufficient dimension.

The slice representations of an isometric Lie group action on a Riemannian manifold of cohomogeneity one or two are orthogonal representations of cohomogeneity one or two, respectively, and hence are polar. Therefore we may immediately rule out all groups contained in a closed subgroup $H \subset L \times L$ which has a non-polar slice representation. Here we can use the classifications [25], [18] and [19] of (hyper)polar actions to exclude many candidates for cohomogeneity two actions. In

fact, the appearance of cohomogeneity two actions is one major technical complication in [19], since for these actions the slice representations do not contain any information about the polarity of the action.

Theorem 18. *Let L be a simply connected exceptional simple compact Lie group and let $H \subset L \times L$ be a closed subgroup acting with cohomogeneity one on L . Then H is conjugate to a group $H_1 \times H_2$ or $H_2 \times H_1$ such that the triple (L, H_1, H_2) occurs in Table 10. In particular, there are no isometric cohomogeneity one actions on E_7 and E_8 and any isometric cohomogeneity one action on E_6 and F_4 is orbit equivalent to a Hermann action.*

L	H_1	H_2	Description
E_6	$SU(6) \cdot Sp(1)$	F_4	E II-IV
E_6	$SU(6) \cdot U(1)$	F_4	
E_6	$SU(6)$	F_4	
E_6	$Spin(10) \cdot U(1)$	F_4	E III-IV
F_4	$Sp(3) \cdot Sp(1)$	$Spin(9)$	F I-II
F_4	$Sp(3) \cdot U(1)$	$Spin(9)$	
F_4	$Sp(3)$	$Spin(9)$	
F_4	$Spin(9)$	$Spin(9)$	F II-II
G_2	$SU(3)$	$SU(3)$	
G_2	$SU(3)$	$SO(4)$	

TABLE 10. Cohomogeneity one actions.

Theorem 19. *Let L be a simply connected exceptional simple compact Lie group and let $H \subset L \times L$ be a closed connected subgroup. Then the action of H on L is of cohomogeneity two if and only if one of the following holds*

- (i) $L = G_2$ and H is conjugate to ΔG_2 ;
- (ii) $L = E_6$ and H is conjugate to a group

$$(Spin(10) \times Spin(10)) \cdot Q \subseteq (Spin(10) \cdot U(1)) \times (Spin(10) \cdot U(1)),$$
 where $Q \subset U(1) \times U(1)$ is a one-dimensional closed connected subgroup such that $Q \neq \Delta U(1)$.
- (iii) H is conjugate to a group $H_1 \times H_2$ or $H_2 \times H_1$ such that the triple (L, H_1, H_2) occurs in Table 11.

Remarks. For convenience, we have formulated the statement of Theorems 18 and 19 only for simply connected groups L ; however, the result of course implies the classification also for the non-simply connected case, since the cohomogeneity of an H -action on L depends only on the conjugacy class of the subalgebra $\mathfrak{h} \subset \mathfrak{l} \oplus \mathfrak{l}$. In the last column of Tables 10 and 11, the types of the symmetric subgroups are given in case of a Hermann action, here we use the notation as in Table 3. Actions which appear in consecutive rows of the tables without separating horizontal lines between them are orbit equivalent to one another.

Proof of Theorems 18 and 19. Let H be a closed connected subgroup of a simply connected exceptional compact Lie group L acting with cohomogeneity one or two.

L	H_1	H_2	polar?	Description
E_6	$\mathrm{Sp}(4)/\{\pm 1\}$	$\mathrm{Spin}(10) \cdot \mathrm{U}(1)$	yes	E I-III
E_6	$\mathrm{Sp}(4)/\{\pm 1\}$	F_4	yes	E I-IV
E_6	$\mathrm{SU}(6) \cdot \mathrm{Sp}(1)$	$\mathrm{Spin}(10) \cdot \mathrm{U}(1)$	yes	E II-III
E_6	$\mathrm{SU}(6) \cdot \mathrm{Sp}(1)$	$\mathrm{Spin}(10)$		
E_6	$\mathrm{Spin}(10) \cdot \mathrm{U}(1)$	$\mathrm{Spin}(10) \cdot \mathrm{U}(1)$	yes	E III-III
E_6	F_4	F_4	yes	E IV-IV
E_6	$\mathrm{Spin}(10)$	F_4	no	
E_6	$\mathrm{S}(\mathrm{U}(5) \cdot \mathrm{U}(1)) \cdot \mathrm{Sp}(1)$	F_4	no	
E_6	$\mathrm{SU}(5) \cdot \mathrm{Sp}(1)$	F_4	no	
E_7	$\mathrm{SO}'(12) \cdot \mathrm{Sp}(1)$	$E_6 \cdot \mathrm{U}(1)$	yes	E VI-VII
E_7	$\mathrm{SO}'(12) \cdot \mathrm{Sp}(1)$	E_6		
F_4	$\mathrm{Spin}(9)$	$\mathrm{Spin}(8)$	no	
F_4	$\mathrm{Spin}(9)$	$\mathrm{Spin}(7) \cdot \mathrm{SO}(2)$	no	
F_4	$\mathrm{Spin}(9)$	$\mathrm{Spin}(6) \cdot \mathrm{Spin}(3)$	no	
F_4	$\mathrm{Spin}(9)$	$G_2^1 \cdot A_1^8$	no	
F_4	$\mathrm{Spin}(9)$	$\mathrm{SU}(3) \cdot \mathrm{SU}(3)$	no	
G_2	$\mathrm{SO}(4)$	$\mathrm{SO}(4)$	yes	G
G_2	$\mathrm{SU}(3)$	$\mathrm{U}(2)$	no	
G_2	$\mathrm{SU}(3)$	$\widehat{\mathrm{U}}(2)$	no	

TABLE 11. Cohomogeneity two actions.

By Lemma 3, H is either contained in ΔL or in $H_1 \times H_2$, where $H_i \subset L$ are maximal connected subgroups. In the first case it follows from the result of Section 3 that the action is of cohomogeneity ≤ 2 if and only if $L = G_2$ and H is conjugate to ΔG_2 .

Subactions of cohomogeneity one Hermann actions. Assume $H \subset H_1 \times H_2$, where $H_i \subset L$ are symmetric subgroups such that the $H_1 \times H_2$ -action on L is of cohomogeneity ≤ 1 . Since there are no transitive actions of this type [21], it follows that the action is a Hermann action of cohomogeneity one, cf. Table 3. (Note that the action F II-II does not appear in Table 3, since the groups H_1 and H_2 are conjugate.) We will treat the subactions of these four actions in the following paragraphs.

F II-II. Assume there is a closed connected subgroup H' of $H = \mathrm{Spin}(9) \times \mathrm{Spin}(9)$ acting with cohomogeneity two on F_4 . The H -orbit $H \cdot e$ through the identity element $e \in F_4$ is the subgroup $\mathrm{Spin}(9) \subset F_4$ and the slice representation of the isotropy group $\Delta \mathrm{Spin}(9)$ at e is equivalent to the 16-dimensional spin representation of $\mathrm{Spin}(9)$. Consider the action of $H'_e = H' \cap \Delta \mathrm{Spin}(9)$ on the invariant subspace $N_0 := N_e(H \cdot e)$ of its slice representation. Now there are two cases, depending on whether this action is transitive on the sphere or not. If H'_e acts transitively on the unit sphere in N_0 , then it follows that $\Delta \mathrm{Spin}(9)$ is contained in H'_e , since there is no non-trivial factorization [21] of $\mathrm{Spin}(9)$. Since $\Delta \mathrm{Spin}(9) \subset \mathrm{Spin}(9) \times \mathrm{Spin}(9)$ is a maximal connected subgroup, it follows that either $H' = \Delta \mathrm{Spin}(9)$, which acts with cohomogeneity 16 or $H' = \mathrm{Spin}(9) \times \mathrm{Spin}(9)$. If H'_e does not act transitively on the unit sphere in N_0 , then it follows that H' acts transitively on $H \cdot e$ and, again, since there is no non-trivial factorization of $\mathrm{Spin}(9)$, it follows that H' is of the form $\mathrm{Spin}(9) \times K$ (or $K \times \mathrm{Spin}(9)$), where $K \subset \mathrm{Spin}(9)$ acts with cohomogeneity

two on the Cayley plane $F_4/\text{Spin}(9)$. These groups have been classified in [25], pp. 172-173.

F II-I. Assume H' is a closed connected subgroup of $H = \text{Spin}(9) \times (\text{Sp}(3) \cdot \text{Sp}(1))$ acting with cohomogeneity one or two on F_4 . From the proof of Theorem 14 we see that it only remains to consider the groups

$$\text{Spin}(9) \times (\text{Sp}(2) \cdot \text{Sp}(1) \cdot \text{Sp}(1)), \quad \text{Spin}(9) \times (\text{Sp}(3) \cdot \text{U}(1)), \quad \text{Spin}(9) \times \text{Sp}(3).$$

The first group acts with cohomogeneity greater than two [25], pp. 172-173, the other groups are known to act with cohomogeneity one [25].

E II-IV. The groups considered in Theorem 14 cannot act with cohomogeneity one or two since their dimension is too small. Thus it remains to consider groups $H = H_1 \times H_2$ where either $H_1 = \text{SU}(6) \cdot \text{Sp}(1)$ or $H_2 = F_4$.

Assume first $H_1 = \text{SU}(6) \cdot \text{Sp}(1)$. The argument in [19], Section 12, p. 478, shows that there is no subgroup $H_2 \subsetneq F_4$ acting with cohomogeneity two on E_6 .

Assume now that $H_2 = F_4$. These actions have been studied in [19], Section 12, pp. 477-478, it is shown there that some of them have non-polar slice representations and thus are of cohomogeneity at least three; furthermore, subactions of cohomogeneity one are determined. In the case where $H_1 = \text{S}(\text{U}(5) \cdot \text{U}(1)) \cdot \text{Sp}(1)$, it is (implicitly) shown in loc. cit. that the action is of cohomogeneity two and that among its subactions $H_1 = \text{SU}(5) \cdot \text{Sp}(1)$ is the only one of cohomogeneity two. If $H = (\text{Sp}(3) \cdot \text{Sp}(1)) \times F_4$ then we may assume $H_1 \subset H_2$ (see [7], p. 200) and the orbit through the identity element $e \in E_6$ is the subgroup F_4 . The slice representation at e is the 26-dimensional irreducible representation of F_4 whose restriction to $\text{Sp}(3) \cdot \text{Sp}(1)$ is non-polar by Proposition 9 (i) and Table 1, hence not of cohomogeneity two. There are no cohomogeneity two subactions since $\dim(H) = 76$.

E III-IV. Let H be a closed connected subgroup of $(\text{Spin}(10) \cdot \text{U}(1)) \times F_4$. Assume first H contains the F_4 -factor. Such actions were studied in [19], Section 12, pp. 478-479 where the cohomogeneity two action of $\text{Spin}(10) \times F_4$ is shown to be non-polar. It is also shown in loc. cit. that all other actions of this type are excluded by a dimension count or have non-polar slice representations. Assume now H contains the factor $\text{Spin}(10) \cdot \text{U}(1)$. By the argument in [19], Section 12, p. 479, we only have to consider the action of $(\text{Spin}(10) \cdot \text{U}(1)) \times \text{Spin}(9)$ which can easily be seen to have a slice representation of cohomogeneity greater than two. In case H does contain neither the F_4 -factor nor $\text{Spin}(10) \cdot \text{U}(1)$, it follows from the last part of the proof of Theorem 14 that the action cannot be of cohomogeneity two.

Subactions of cohomogeneity two Hermann actions. The orbit equivalent subactions of Hermann actions of cohomogeneity two were determined in the proof of Theorem 13.

Other actions. It remains to determine those cohomogeneity two actions which are neither subactions of Hermann actions nor of σ -actions.

Actions on G_2 . For actions on G_2 , the classification follows from Theorem 17.

Actions on F_4 . Adding up dimensions of the groups given in Table 6, we see that the only actions of a group of sufficient dimension are (1)–(3) and (1)–(4). It is shown in Section 6 that both actions are non-polar and of cohomogeneity two and it follows from the arguments given there that there are no proper subactions of cohomogeneity two.

Actions on E_6 . Again by counting dimensions we only need to consider the action (5)–(9). Since a slice representation of this action is non-polar, it follows that the cohomogeneity is greater than two.

Actions on E_7 . We only need to consider the action (11)–(14), which has a non-polar slice representation.

Actions on E_8 . Since the only closed connected subgroups in E_8 of dimension ≥ 110 are symmetric and Hermann actions on E_8 are of cohomogeneity ≥ 4 , we conclude that there are no isometric actions on E_8 of cohomogeneity one or two. \square \square

13. LOW COHOMOGENEITY ACTIONS ON E_8

Theorem 20. *Let $H \subset E_8 \times E_8$ be a closed connected subgroup acting on E_8 . Then the H -action on E_8 is of cohomogeneity k with $0 < k < 20$ if and only if it is conjugate to the action of one of the groups given in Table 12.*

Subgroup of $E_8 \times E_8$	Range	Cohomogeneity
ΔE_8		8
$(E_7 \times E_7) \cdot Q$	$Q \subseteq \mathrm{Sp}(1) \times \mathrm{Sp}(1)$	$10 - \dim Q$
$\mathrm{SO}'(16) \times \mathrm{SO}'(16)$		8
$(E_7 \cdot P) \times \mathrm{SO}'(16)$	$P \subseteq \mathrm{Sp}(1)$	$7 - \dim P$
$(E_7 \cdot P) \times \mathrm{Spin}(15)$	$P \subseteq \mathrm{Sp}(1)$	$10 - \dim P$

TABLE 12. Low cohomogeneity actions on E_8 .

Proof. Assume $H \subset E_8 \times E_8$ is a closed connected subgroup acting non-transitively and with cohomogeneity ≤ 19 on E_8 . We know from Lemma 3 that H either is contained in ΔE_8 or in a group of the form $H \subseteq H_1 \times H_2$, where $H_i \subset E_8$ are maximal closed connected subgroups. In the first case it follows that $H = \Delta E_8$ since the maximal dimension of a proper closed subgroup of E_8 is 136. In the latter case, it follows that $\dim H_i \geq 93 = \dim E_8 - \dim E_7 \cdot \mathrm{Sp}(1) - 19 = 248 - 136 - 19$. We can see from [7] that the only such maximal connected subgroups of E_8 are the two symmetric subgroups $E_7 \cdot \mathrm{Sp}(1)$ and $\mathrm{SO}'(16)$. The only connected subgroups of these two groups whose dimension is greater than 92 are $E_7 \cdot \mathrm{U}(1)$, $E_7 \subset E_7 \cdot \mathrm{Sp}(1)$ and $\mathrm{Spin}(15) \subset \mathrm{SO}'(16)$, respectively. Using the data of the slice representation of the Hermann action E VIII-IX as given by Table 4 and looking at the isotropy representations of the symmetric spaces E VIII and E IX we can now verify the content of Table 12.

Consider closed connected subgroups H of $(E_7 \cdot \mathrm{Sp}(1)) \times (E_7 \cdot \mathrm{Sp}(1))$. Every such subgroup of dimension greater than 228 is of the form $(E_7 \times E_7) \cdot Q$, where Q is a closed connected subgroup of $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$. The principal isotropy algebra for the action of $(E_7 \times E_7) \cdot Q$ on E_8 is isomorphic to $\mathfrak{spin}(8)$ according to [15], p. 199; hence the cohomogeneity of this action is $10 - \dim Q$. The cohomogeneity of the Hermann action of $\mathrm{SO}'(16) \times \mathrm{SO}'(16)$ on E_8 is equal to $\mathrm{rk}(E_8/\mathrm{SO}'(16)) = 8$. Now consider subactions of the Hermann action E VIII-IX. Every subgroup of $(E_7 \cdot \mathrm{Sp}(1)) \times \mathrm{SO}'(16)$ of dimension greater than 228 is of the form $(E_7 \cdot P) \times \mathrm{SO}'(16)$ or $(E_7 \cdot P) \times \mathrm{Spin}(15)$, where $P \subseteq \mathrm{Sp}(1)$ is a closed connected subgroup. By Table 4, an isotropy subgroup of the Hermann action E VIII-IX is locally isomorphic to $\mathrm{U}(8)$ and its slice representation is on the Lie algebra level equivalent to the isotropy representation of $\mathrm{SO}(16)/\mathrm{U}(8)$. Following [15], the principal isotropy subalgebra is $\mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)$. This shows that the principal isotropy group is

12-dimensional for all choices of $P \subseteq \mathrm{Sp}(1)$ since we have the inclusion $4 \cdot \mathfrak{su}(2) \subset \mathfrak{su}(8) \subset \mathfrak{e}_7$. Finally, consider the action of $(E_7 \cdot \mathrm{Sp}(1)) \times \mathrm{Spin}(15)$ on E_8 . It follows from the argument above that one isotropy subgroup is locally isomorphic to $\mathrm{U}(7)$. The isotropy representation of $\mathrm{SO}(16)/\mathrm{U}(8)$ restricted to $\mathrm{U}(7)$ splits into $\Lambda^2 \mathbb{C}^7 \oplus \mathbb{C}^7$ and has finite principal isotropy subgroups. This shows that the actions in the last row of Table 12 also have finite principal isotropy subgroups. \square \square

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